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Cooling of a Heat-Generating Strip Immersed in a Laminar Channel Flow

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An asymptotic and numerical analysis was conducted for the cooling of a heated strip immersed in a laminar channel flow. The strip material, which can represent an electronic chip, generates heat internally at a uniform rate. Taking into account the finite thermal conductivity of the strip, the nondimensional temperature profile in the strip, the maximum longitudinal temperature differences, and the corresponding Nusselt number have been obtained as functions of the nondimensional parameters α and *Pe*. The heat conduction parameter α represents the competition between the longitudinal heat conduction in the strip and the heat convection in the laminar cooling flow. The parameter *Pe* is the well-known Peclet number of the problem. The numerical and asymptotic results for these show a strong dependence of the parameters α and *Pe*. Therefore, the proposed cooling mechanism serves to identify the role of longitudinal heat conduction effects in the strip. This mechanism can seriously affect the thermal performance of this simplified model of an electronic circuit board.

Nomenclature

$B(l, n)$	=	beta function
C	=	specific heat of the laminar cooling flow
H	=	channel width
h	=	thickness of the strip
L	=	length of the strip
Nu	=	Nusselt number
Pe	=	Peclet number of the laminar cooling flow
Pr	=	Prandtl number of the laminar cooling flow
q	=	volumetric heat production
Re	=	Reynolds number of the laminar cooling flow
T_F	=	fluid temperature
T_w	=	strip temperature
$\bar{T}_w$	=	average temperature on the strip
ΔT_{wm}	=	maximum difference of the strip temperature, $T_w(x = L) - T_w(x = 0)$
T_∞	=	freestream temperature of the laminar cooling flow
u	=	velocity profile of the laminar cooling flow
x, y	=	Cartesian coordinates
z	=	nondimensional normal coordinate of the strip
α	=	heat conduction parameter
δ	=	thickness of the thermal boundary layer in the laminar cooling flow
ε	=	aspect ratio of the strip
θ_w	=	nondimensional temperature of the strip
$\bar{\theta}_w$	=	average nondimensional temperature of the strip
λ	=	thermal conductivity of the fluid
λ_w	=	thermal conductivity of the strip
ν	=	kinematic viscosity of the fluid
ξ	=	nondimensional longitudinal coordinate defined in Eq. (7)
ρ	=	density of the fluid
ρ_w	=	density of the strip
χ	=	nondimensional longitudinal coordinate defined in Eq. (6)

Subscripts

l	=	conditions at the leading edge of the strip
w	=	conditions at the strip
∞	=	conditions in the laminar cooling flow

I. Introduction

THE electronic cooling analysis of small heat-generating strips or chips has been recognized in the specialized literature as an active and fundamental research area, due to the influence of this factor on the control of the electrical efficiency of different types of board circuitry. Xu and Guo[1] showed that to maintain the device junction temperature below a maximum limit, the increase of volumetric heat production rates must be controlled. Otherwise, the temperature differences for conventional systems affect the component's efficiency and these overestimated chip temperature gradients can introduce thermal failure between the components of the board circuitry, changing the electronic performance drastically. In some cases these thermal failures can generate irreversible mechanical fractures. Therefore, the theoretical analysis of thermal conditions on electronic package surfaces is very important. In general, these conditions are unknown and for a given heat generation rate, the prediction of temperature profiles at the electronic chip is of primordial importance for obtaining high performance of the involved electronic components.

The foregoing fundamental and practical aspects offer an excellent opportunity to explore systematically this class of conjugated heat transfer models. Here, we accept that the cooling effect is directly related to the volumetric heat production rate, and as a consequence, the conjugate heat transfer formulation is inevitable. The fundamental importance of these thermal interactions between forced and natural convection flows and thermal sources on surfaces, has been pointed out by Incropera[2] and Jaluria.[3] Later, Sathe

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and Joshi[4] showed the importance of coupled heat transfer between a heat-generating substrate-mounted protrusion and a liquid-filled two-dimensional enclosure. For simplicity, the flush heaters in these works were idealized as uniform heat sources. Méndez and Treviño[5] analyzed the conjugated heat transfer between a natural convection flow and an embedded vertical strip in a substrate with nonuniform heat generation rate. They used numerical and analytical perturbation techniques to clarify the role of longitudinal heat transfer effects on a vertical thin plate in a natural convective cooling process. Sometimes, passive cooling by natural convection is preferred, because it is characterized by simplicity of design, absence of noise, and high reliability. However, for increasing volumetric heat generation rates, other cooling techniques are needed, as pointed out by Tou et al.[6] and Sun et al.[7] Several works have appeared in the literature analyzing the electronic cooling chip problem with forced flow configurations. Ramadhyani et al.[8] theoretically, Ortega et al.[9] experimentally, and Incropera et al.[10] using experimental and theoretical techniques considered the problem of conjugate heat transfer from discrete heat sources mounted on a wall of a channel exposed to fully developed laminar flow. Following a simpler physical model, Rizk et al.[11] deduced an analytical solution for the conjugate heat transfer problem of a flow past a heated block. A well-documented state of the art can be found in Cole,[12] who clarifies the essential role of electronic cooling, taking advantages of scaling laws and involved nondimensional parameters. A more complex situation has been considered recently by Chuang et al.[13] in studying numerically the heat transfer between a three-dimensional rectangular duct and heat-generating chips, showing that higher inlet velocity leads to heat transfer enhancement in the internal region of the duct.

In this work, with the aid of perturbation as well as numerical techniques, we obtain the temperature profile of a thin strip with uniform internal heat generation. The heat-generation strip is embedded in an idealized adiabatic substrate and is permanently cooled in a rectangular channel laminar flow. We follow the basic ideas developed by Rizk et al.[11] and Cole[12] to understand this conjugate heat transfer process. Thus, recognizing that previous authors[8–10] took into account the confined character of the flow pattern to analyze the thermal interaction between the cooling flow and the heated strip, we note basic differences between those models and the present formulation: 1) The influence of the longitudinal heat conduction effects in the cooling flow was taken into account by those authors. In our case, we propose a thermal boundary layer approximation, where longitudinal heat conduction effects are negligible. 2) We solve the conjugate heat transfer problem between the cooling flow and the heated strip, whereas these authors studied the conjugate heat transfer between the flow and the substrate (and insulation), assuming isothermal heat sources or strips. On the other hand, the basic coincidence with the previous authors is to assume a fully developed velocity profile in the rectangular channel. In the present work, the heat transfer to the laminar cooling flow is evaluated with the aid of thermal boundary layer theory. Therefore, a forced boundary layer develops, causing conjugated heat transfer between the chip and the cooling flow, because the temperature of the strip is not known in advance. The analytical and numerical results serve to establish a direct correlation between strip temperature profiles, uniform heat-generation rates, and the laminar cooling flow. This relationship is clearly shown using appropriate nondimensional parameters.

II. Formulation

The physical model under study is shown in Fig. 1. In the Cartesian coordinate system, the upper left corner of the strip coincides with the origin, whose y axis points outward in the direction normal to the strip and whose x axis points outward in the strip's longitudinal direction. The heat-conducting strip of length L and thickness h is embedded in a rectangular channel of width H. Due to internal heat generation in the strip with a uniform volumetric rate q, an important fraction of the heat transfer occurs between the strip and the laminar cooling flow, because the walls of this channel where the heated strip is located are composed of an adiabatic substrate. Thus we assume that the lower, right, and left faces are adiabatic. To fulfill these thermal conditions, the ratio of the thermal conductivity

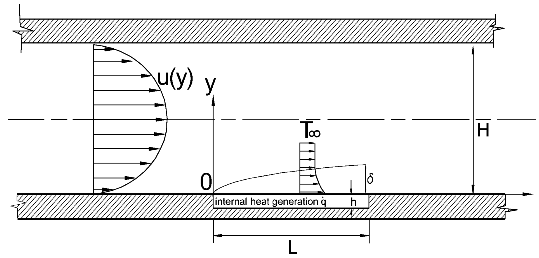

Fig. 1 Schematic diagram of the studied physical model.

of the substrate to the thermal conductivity of the strip is assumed to be vanishingly small compared with unity. There are many well-documented practical situations that reflect this case and can be found elsewhere.[4]

On the other hand, the upper face of the strip contacts a laminar cooling flow with a well-known fully developed velocity profile given by

$$u = 6\bar{u}\left[\left(\frac{y}{H}\right) - \left(\frac{y}{H}\right)^2\right], \qquad \text{with} \qquad \bar{u} = -\frac{1}{12\mu}\frac{\mathrm{d}P}{\mathrm{d}x}H^2 \quad (1)$$

where $\bar{u}$ represents the mean value of the velocity.

To obtain appropriate scales for the problem, we use an order of magnitude analysis of the energy equation of the cooling fluid to show that the thickness of the thermal boundary layer is related to the length of the strip by[14]

$$\delta/L \sim [(H/L)^2(1/6Pe)]^{\frac{1}{3}} \quad (2)$$

where $Pe = RePr$ and $Re = \bar{u}H/\nu$. The variables ρ and μ are the density and the dynamic viscosity of the fluid, respectively. On the other hand, a global balance of thermal energy establishes that the heat flux from the strip to the cooling flow can be represented by

$$qh \sim \lambda_w(\Delta T_w/h) \sim \lambda(\Delta T_F/\delta) \quad (3)$$

where ΔT_w is the characteristic temperature drop in the transverse direction of the wall, and ΔT_F is also the characteristic temperature drop in the cooling fluid. From Eq. (3), the first term corresponds to the thermal energy generated in the strip, whereas the third one is the heat transferred to the cooling fluid. Combining Eqs. (2) and (3) yields

$$\Delta T_F \sim c(qh/\lambda)(H^2L/6Pe)^{\frac{1}{3}} \qquad \text{and} \qquad \Delta T_w \sim qh^2\big/\lambda_w \quad (4)$$

Using these relationships, we also obtain that

$$\Delta T_F/\Delta T_w \sim \alpha/\varepsilon^2$$

$$\text{with} \qquad \alpha = c(\lambda_w/\lambda)(h/L)[(H/L)^2(1/6Pe)]^{\frac{1}{3}} \quad (5)$$

where ε is the aspect ratio of the strip, $\varepsilon = h/L$, assumed to be very small compared with unity, and c is a constant of order unity to be given later. Parameter α is the nondimensional longitudinal heat conductance of the strip and is a measure of the longitudinal heat conduction effects in the strip on the cooling process. Cole[12] and Stein et al.[15] obtained similar expressions to define this nondimensional conjugate parameter. However, the Peclet number involved in Eq. (5) is based on the width, H, of the rectangular channel, whereas the Peclet number derived by Cole is defined with the aid of the half-length of heated strip. On the other hand, Stein used practically the same conjugate parameter as Cole (in this case, the Peclet number is defined with the length of the heated source) and called it the Cole number, Co. The fundamental discrepancy between those authors and the present definition of α is that the geometry used by Cole and Stein corresponds to an external flow on a heated strip, whereas we use the rectangular channel. This parameter can assume any value depending on the geometrical and physical properties of the strip and the fluid. For instance, from the order relationship, Eq. (5)

with $\alpha/\varepsilon^2 \gg 1$, the transverse temperature variations of the strip are very small compared with the temperature differences in the fluid, that is, $\Delta T_w \ll \Delta T_F$. In a previous work,[16] we named this limit the thermally thin wall regime. For values of $\alpha/\varepsilon^2 \sim 1$, the temperature variations in both directions of the strip are very important and of the same order of magnitude as the temperature differences in the fluid. This regime is called the thermally thick wall regime. In this limit and with $\varepsilon \ll 1$, the longitudinal heat conduction through the strip is very small and can be neglected.

With the above discussion, we introduce the following nondimensional variables for the strip,

$$\chi = x/L, \qquad z = (y+h)/h, \qquad \theta_w = (T_w - T_\infty)/\Delta T_F$$
$$\text{with} \qquad \Delta T_F = c(qh/\lambda)(H^2 L/6Pe)^{\frac{1}{3}} \tag{6}$$

and for the fluid,

$$\xi = y/\delta, \qquad \theta = (T - T_\infty)/\Delta T_F \tag{7}$$

Therefore, the nondimensional energy balance equations for the strip and fluid take the forms

$$\alpha \frac{\partial^2 \theta_w}{\partial \chi^2} + \frac{\alpha}{\varepsilon^2}\frac{\partial^2 \theta_w}{\partial z^2} + 1 = 0 \tag{8}$$

and

$$\xi \frac{\partial \theta}{\partial \chi} - \left(\frac{L}{H}\frac{1}{6Pe}\right)^{\frac{1}{3}} \xi^2 \frac{\partial \theta}{\partial \chi} = \frac{\partial^2 \theta}{\partial \xi^2} + \left[\left(\frac{H}{L}\right)^2 \frac{1}{6Pe}\right]^{\frac{2}{3}} \frac{\partial^2 \theta}{\partial \chi^2} \tag{9}$$

Equation (9) is the energy equation for the fluid, retaining the heat-conduction terms (right-hand side) in both directions. However, in the present analysis we use the asymptotic form of this equation, taking into account finite values of the aspect ratio H/L and $Pe \gg 1$. We use this limit because the majority of the practical applications occur under this approximation. Thus, taking the limit of $Pe \gg 1$ and finite values of the aspect ratio H/L, we simplify the above equation by this other,

$$\xi \frac{\partial \theta}{\partial \chi} = \frac{\partial^2 \theta}{\partial \xi^2} \tag{10}$$

This approximation means that the heat transfer from the strip to the cooling flow occurs only in a thin thermal boundary layer adjacent to the upper face of the strip. In this case, the corresponding velocity profile within this layer appears, in a first approximation, as a linear profile,[2,12,15–17] which in fact is represented by the factor ξ that multiplies the left-hand side of Eq. (10). Therefore, the above approximation with finite values of H/L and $Pe \gg 1$ makes it possible to neglect the second term of the left-hand side of Eq. (9), which represents a higher-order correction. In addition, this same approximation makes it possible to neglect the conductive-heat term in the longitudinal direction. The corresponding boundary conditions are, for the strip,

$$\frac{\partial \theta_w}{\partial \chi} = 0 \quad \text{at} \quad \chi = 0, 1 \quad \text{and} \quad \frac{\partial \theta_w}{\partial z} = 0 \quad \text{for} \quad z = 0 \tag{11}$$

and for the fluid

$$\theta(\chi, \xi \to \infty) = \theta(\chi = 0, \xi) = 0 \tag{12}$$

To complete the boundary conditions, we need an additional condition at $z = 1$ ($\xi = 0$), which corresponds to demanding continuity of the temperatures and heat transfer rates,

$$\theta(\chi, \xi = 0) = \theta_w(\chi) \qquad \text{and} \qquad \left.\frac{\partial \theta_w}{\partial z}\right|_{z=1} = \frac{\varepsilon^2}{\alpha}\left.\frac{\partial \theta}{\partial \xi}\right|_{\xi=0} \tag{13}$$

where the term $\partial\theta/\partial\xi|_{\xi=0}$ represents the nondimensional heat flux from the upper surface of the strip to the cooling fluid. To obtain the solution of the energy equation, Eq. (10), we propose quasi-similarity variables given by

$$\theta = \theta_w \varphi \qquad \text{and} \qquad \zeta = \xi/\chi^{\frac{1}{3}} \tag{14}$$

thus Eq. (10) reduces to

$$\frac{\partial^2 \varphi}{\partial \zeta^2} + \frac{1}{3}\zeta^2 \frac{\partial \varphi}{\partial \zeta} = \chi \zeta \frac{\partial \varphi}{\partial \chi} \tag{15}$$

with the corresponding boundary conditions

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi(\zeta \to \infty) = 0 \tag{16}$$

where as Eq. (13) can be written as

$$\left.\frac{\partial \theta_w}{\partial z}\right|_{z=1} = -\frac{\varepsilon^2}{\alpha}\frac{1}{\chi^{\frac{1}{3}}}\left\{\theta_{wl} + \int_{\theta_{wl}}^{\theta_w} K(\chi, \chi')\,\mathrm{d}\theta_w'\right\} \tag{17}$$

where this last boundary condition, Eq. (17), contains the solution of Eq. (15), which is obtained with the Lighthill approximation.[17] This integral approach takes into account that the longitudinal velocity component u is still proportional to the transverse distance y when the temperature field is confined inside the zone of the velocity field, and this occurs frequently for large values of the Prandtl number. However, this approximation gives excellent results even for Prandtl numbers of order unity. In particular, the kernel in Eq. (17) is given by

$$K(\chi, \chi') = (1 - \chi'/\chi)^{-\frac{1}{3}} \tag{18}$$

and θ_{wl} corresponds to the value of the nondimensional temperature at the leading edge of the strip. The assumed value for the constant c in the definition of α is $c = 3^{1/3}/\Gamma(1/3)$. In general, Eqs. (8), (11), and (17) must be numerically integrated. In the following section, we obtain asymptotic solutions for the thermally thin and thick wall regimes. We also include the numerical solution for the thermally thin wall regime.

III. Thermally Thin Wall Regime ($\alpha/\varepsilon^2 \gg 1$)

For values of α/ε^2 very large compared with unity, the relationship (5) dictates that the temperature variations in the normal direction of the strip can be neglected. Therefore, in a first approximation, θ_w only depends on the coordinate χ. In this regime, the nondimensional energy equation for the strip, Eq. (8), can be integrated along the normal coordinate, and after applying the appropriate boundary conditions (11) and (17), we obtain

$$\alpha \frac{\mathrm{d}^2 \theta_w}{\mathrm{d}\chi^2} = -1 + \frac{1}{\chi^{\frac{1}{3}}}\left\{\theta_{wl} + \int_{\theta_{wl}}^{\theta_w} K(\chi, \chi')\,\mathrm{d}\theta_w'\right\} \tag{19}$$

This equation must be solved with the adiabatic conditions for the lateral surfaces of the strip given by Eqs. (11). In the following sections we present asymptotic solutions for very large values of α, that is, $\alpha \gg 1$, and for values of $\alpha \to 0$, respectively. In both cases, we accept that the limit $\alpha/\varepsilon^2 \gg 1$ is valid. In addition, we obtain the numerical solutions for these limits in order to compare them with previous analytical solutions. Thus, we solve Eq. (19) using numerical techniques reported elsewhere.[16]

A. Asymptotic Limit $\alpha \gg 1$

The asymptotic limit of $\alpha \to \infty$ is regular and the solution can be obtained with the aid of a regular perturbation technique, using the inverse of α as the small parameter of expansion. For very large values of the parameter, the nondimensional temperature of the plate, θ_w, changes very little in the longitudinal direction (of order α^{-1}).

Therefore, we assume that the nondimensional temperature of the strip can be expanded as

$$\theta_w = \sum_{i=0}^{\infty} \frac{1}{\alpha^i}\theta_{wi}(\chi) \tag{20}$$

Introducing the above relationship (20) into the nondimensional governing equation for the strip (19), we obtain, after collecting terms of the same power of α, the set of equations

$$\frac{d^2\theta_{w0}}{d\chi^2} = 0 \tag{21}$$

$$\frac{d^2\theta_{w1}}{d\chi^2} = -1 + \frac{1}{\chi^{\frac{1}{3}}}\left[\theta_{0l} + \int_{\theta_{0l}}^{\theta_0} K(\chi,\chi')\, d\theta'_{w0}\right] \tag{22}$$

$$\frac{d^2\theta_{w(i+1)}}{d\chi^2} = \frac{1}{\chi^{\frac{1}{3}}}\left[\theta_{il} + \int_{\theta_{il}}^{\theta_i} K(\chi,\chi')\, d\theta'_{wi}\right] \qquad \text{for} \qquad i = 1, 2, 3 \tag{23}$$

with the adiabatic boundary conditions

$$\left.\frac{d\theta_{wi}}{d\chi}\right|_{\chi=0,1} = 0, \qquad \text{for} \qquad i = 0, 1, 2, \ldots \tag{24}$$

Integration of Eq. (21) with the corresponding boundary conditions (24) gives $\theta_{w0} = C_0$, where C_0 is an integration constant and must be determined by solving the next higher order equation, Eq. (22). Integrating Eq. (22) in the form

$$\int_0^1 d\chi$$

and assuming adiabatic conditions at both edges of the strip, yields $C_0 = \frac{2}{3}$. This procedure can be used to obtain the solution to higher orders. Therefore, the solution to Eq. (22) is given by

$$\theta_{w1}(\chi; m) = -\frac{\chi^2}{2} + \frac{3}{5}\chi^{\frac{5}{3}} + C_1 \tag{25}$$

where C_1 is a constant given as

$$C_1 = \frac{2}{3}\left[\frac{3}{8}B\left(2, \frac{2}{3}\right) - \frac{3}{7}B\left(\frac{5}{3}, \frac{2}{3}\right)\right] \tag{26}$$

Following the same procedure, the second-order solution is

$$\theta_2(\chi; m) = \left\{\frac{3}{5}\left[\frac{3}{8}B\left(2, \frac{2}{3}\right) - \frac{3}{7}B\left(\frac{5}{3}, \frac{2}{3}\right)\right]\right\}\chi^{\frac{5}{3}} - \frac{9}{88}B\left(2, \frac{2}{3}\right)\chi^{11/3} + \frac{9}{70}B\left(\frac{5}{3}, \frac{2}{3}\right)\chi^{10/3} + C_2 \tag{27}$$

with C_2 given as

$$C_2 = \frac{2}{3}\left[-\frac{9}{14}C_1 B\left(\frac{5}{3}, \frac{2}{3}\right) + \frac{9}{104}B\left(2, \frac{2}{3}\right)B\left(\frac{11}{3}, \frac{2}{3}\right) - \frac{3}{28}B\left(\frac{5}{3}, \frac{2}{3}\right)B\left(\frac{10}{3}, \frac{2}{3}\right)\right] \tag{28}$$

Therefore, up to terms of second order, the nondimensional plate temperature is given by

$$\theta_w = \frac{2}{3} + \frac{1}{\alpha}\left(-\frac{\chi^2}{2} + \frac{3}{5}\chi^{\frac{5}{3}} + C_1\right) + \frac{1}{\alpha^2}\left\{\left\{\frac{3}{5}\left[\frac{3}{8}B\left(2, \frac{2}{3}\right) - \frac{3}{7}B\left(\frac{5}{3}, \frac{2}{3}\right)\right]\right\}\chi^{\frac{5}{3}} - \frac{9}{88}B\left(2, \frac{2}{3}\right)\chi^{11/3} + \frac{9}{70}B\left(\frac{5}{3}, \frac{2}{3}\right)\chi^{10/3} + C_2\right\} \tag{29}$$

and the averaged nondimensional temperature, up to terms of order $1/\alpha^2$ is then given by

$$\bar{\theta}_w = \int_0^1 \theta_w\, d\chi = \theta_{w0} + \frac{1}{\alpha}\bar{\theta}_{w1} + \frac{1}{\alpha^2}\bar{\theta}_{w2} + \cdots = \frac{2}{3} + \frac{1}{\alpha}\left(-\frac{1}{6} + \frac{9}{40} + C_1\right) + \frac{1}{\alpha^2}\left[\frac{27}{80}C_1 - \frac{27}{1232}B\left(2, \frac{2}{3}\right) + \frac{27}{910}B\left(\frac{5}{3}, \frac{2}{3}\right) + C_2\right] \tag{30}$$

B. Asymptotic Limit $\alpha \to 0$

In the thermally thin wall regime, we have the limiting case of $\alpha \to 0$, but with $\alpha/\varepsilon^2 \gg 1$. The longitudinal heat conduction in the strip is very small and can be neglected except in regions close to the edges of the plate, where local thermal boundary layers appear. However, the structure of these regions has only a local influence. The procedure to analyze this limit is similar to the technique developed in the classical work of Levéque[18] for studying the cooling of a surface submerged in an external laminar flow. Basically, this scheme of solution provides the relationship between the shear stress and the heat transfer to the cooling flow. In a sense, the integral formulation reported by Lighthill[17] generalizes the Levéque result. However, in our case, the internal heat generation is known and is given by the left-hand side of Eq. (31), representing an integral equation to determine the temperature of the upper surface of the strip. From Eq. (19) with $\alpha = 0$, we obtain

$$1 = \frac{1}{\chi^{\frac{1}{3}}}\int_0^{\chi} K(\chi,\chi') \qquad \frac{d\theta'_{w\alpha 0}}{d\chi'}\, dx' \tag{31}$$

The solution of the foregoing equation can easily be obtained with the aid of Abel's integral transform. The nondimensional temperature of the strip, $\theta_w(\chi)$, is given by

$$\theta_{w\alpha 0} = \left[3 \Big/ B\left(\frac{1}{3}, \frac{2}{3}\right)\right]\chi^{\frac{1}{3}} \tag{32}$$

and the nondimensional average temperature is

$$\bar{\theta}_{w\alpha 0} = 9 \Big/ 4B\left(\frac{1}{3}, \frac{2}{3}\right) \tag{33}$$

IV. Thermally Thick Wall Limit ($\alpha/\varepsilon^2 \ll 1$)

In this regime, the longitudinal heat conduction is also very small and is to be neglected. A similar conjugate heat transfer problem was previously analyzed by the pioneering work of Luikov,[19] using an external laminar boundary layer flow on a flat plate. This author assumes a linear temperature distribution in the plate, whereas in our case, we anticipate a parabolic temperature profile due to uniform heat generation. Thus, in the present work, we have a developed laminar flow in a rectangular channel, where the effect of a heated source or strip embedded in a substrate causes a developing thermal boundary layer. Following the basic ideas developed by Luikov, we present the following analysis. The energy balance equation for the strip then reduces to

$$\frac{\partial^2\theta_w}{\partial z^2} = -\frac{\varepsilon^2}{\alpha} \tag{34}$$

Equation (34) has to be solved with the boundary conditions

$$\left.\frac{\partial\theta_w}{\partial z}\right|_{z=0} = 0 \tag{35}$$

and

$$\left.\frac{\partial\theta_w}{\partial z}\right|_{z=1} = -\frac{\varepsilon^2}{\alpha}\frac{1}{\chi^{\frac{1}{3}}}\int_0^{\theta_w}\left[1 - \frac{\chi'}{\chi}\right]^{-\frac{1}{3}}\frac{d\theta'_w}{d\chi'}\, d\chi' \tag{36}$$

Integrating Eq. (34) in the normal direction, and using both conditions (35) and (36), we obtain that

$$\frac{\partial \theta_w}{\partial z} = \frac{\varepsilon^2}{\alpha}\left\{1 - \frac{1}{\chi^{\frac{1}{3}}}\int_0^{\chi}\left[1-\frac{\chi'}{\chi}\right]^{-\frac{1}{3}}\frac{\mathrm{d}\theta_w'}{\mathrm{d}\chi'}\,\mathrm{d}\chi' - z\right\} \tag{37}$$

Therefore, the nondimensional temperature of the strip is

$$\theta_w = \theta_{wu} + (\varepsilon^2/2\alpha)(1-z^2) \tag{38}$$

where θ_{wu} is the nondimensional temperature at the upper surface of the strip, given by

$$\theta_{wu} = \left[3\Big/B\left(\frac{1}{3},\frac{2}{3}\right)\right]\chi^{\frac{1}{3}} \tag{39}$$

and is exactly the same as that obtained for the thermally thin wall regime. The nondimensional average temperature is then

$$\bar{\theta}_w = 9\Big/4B\left(\frac{1}{3},\frac{2}{3}\right) + \frac{1}{3}\frac{\varepsilon^2}{\alpha} \tag{40}$$

In the limit of $\varepsilon^2/\alpha \to 0$, the total thermal energy of the strip in this regime is exactly the same as for the case of $\alpha \to 0$, that is, the thermally thin wall regime given by Eq. (33).

V. Results and Discussion

In all numerical calculations estimated in this work, we use the following data: $T_\infty = 300$ K, $\lambda_w/\lambda = 100$, $Pr = 0.7$, and three values of the volumetric heat generation rates, $q = 40$, 80, and 120 kW/m^3. The length of the strip was 5 cm, its thickness was 1 cm, and the distance between the plates was $H = 2$ cm. To validate the analytical results, Eq. (19) for the thermally thin wall regime was integrated numerically, together with the adiabatic boundary conditions, Eqs. (11), using the Runge–Kutta technique described elsewhere.[16] Specifically, the boundary value problem is transformed to an initial value problem with the given initial conditions for the nondimensional temperature and its gradient. Therefore, a conventional shooting-iteration method was applied, due to the unknown value of θ_{wl} for each value of the parameter α.

Figures 2 and 3 show different numerical and asymptotic results for the nondimensional temperature distribution θ_w as a function of the nondimensional coordinate χ in the thermally thin wall regime

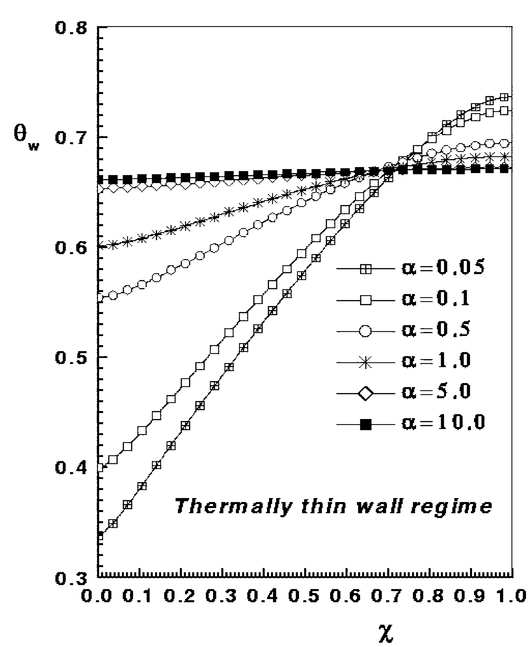

Fig. 2 Numerical solution (for the thermally thin wall regime) of the nondimensional temperature of the strip, θ_w, as a function of the nondimensional longitudinal coordinate χ for different values of the nondimensional conduction parameter α.

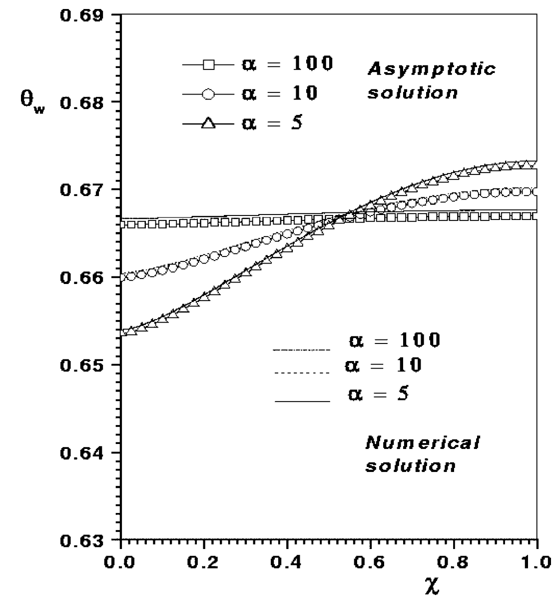

Fig. 3 Numerical solution (for the thermally thin wall regime) of the nondimensional temperature of the strip, θ_w, as a function of the nondimensional longitudinal coordinate χ for different values of the nondimensional conduction parameter α. The analytical solution for the thermally thin wall regime given by Eq. (29) is also plotted.

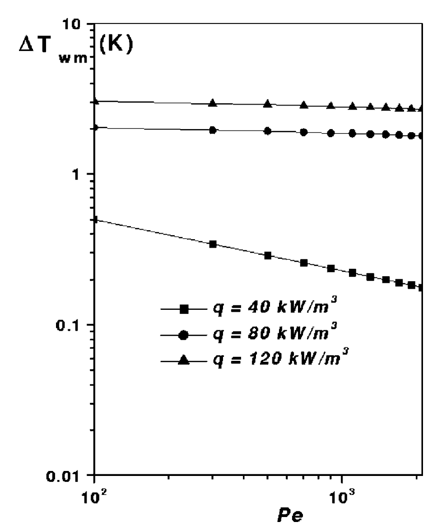

Fig. 4 Maximum difference of the strip temperature, ΔT_{wm}, as a function of the Peclet number, *Pe*, and different values of the internal heat generation rate, *q*.

and different values of the parameter α. In both figures, the temperature is almost flat for values of $\alpha > 5$. In Fig. 2, we have only plotted the numerical solutions and for smaller values of α, the nondimensional temperature θ_w decreases strongly at the leading edge of the strip and increases at the trailing edge. Due to this dependence of θ_w on α, it is really important to take care of the corresponding longitudinal temperature gradients. Therefore, the criterion for operation with better electrical performance depends on which tolerance is accepted: a strip working with large nondimensional longitudinal temperature differences (decreasing values of α) and a low average temperature $\bar{\theta}_w$ or a strip that can attenuate the nondimensional longitudinal temperature differences with high average temperature $\bar{\theta}_w$. To clarify this aspect, we present lines below other numerical results. Additionally, in Fig. 3, we show a comparison between asymptotic and numerical solutions of the temperature θ_w for three values of the nondimensional parameter $\alpha(=5, 10$, and 100). This figure was plotted to validate our numerical scheme. In particular, for the selected values of α, the comparison is very good.

Following the above comments, in Fig. 4 we show, for the thermally thin wall regime, the maximum difference of the strip

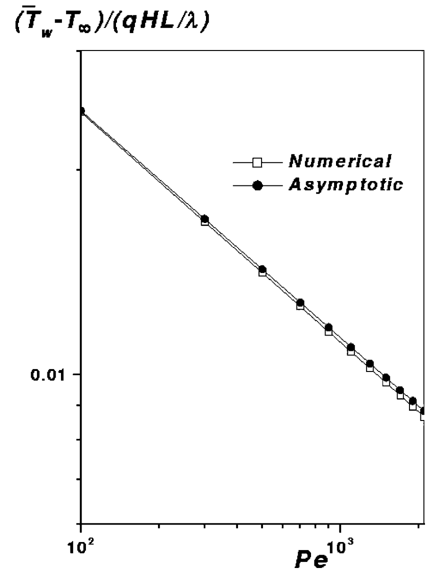

Fig. 5 Comparison between the numerical and asymptotic solutions of the nondimensional average temperature differences, ($T_w - T_\infty$)/(qHL/λ), as a function of the Peclet number, *Pe*.

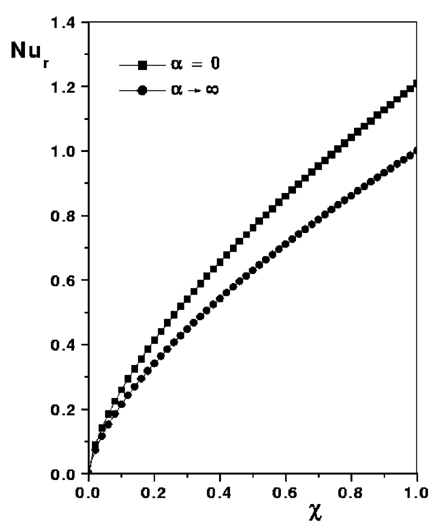

Fig. 6 Analytical comparison of the reduced local Nusselt number *Nu*, as a function of the longitudinal coordinate χ for the asymptotic limits $\alpha = 0$ and $\alpha \gg 1$.

temperature $\Delta T_{wm} = T_w(\chi = 1) - T_w(\chi = 0)$ as a function of the Peclet number Pe. For increasing values of the Peclet number (or decreasing values of α; see Eq. (5)), ΔT_{wm} is shown to decrease. This behavior is more pronounced for decreasing values of the volumetric heat generation rates q. To interpret this behavior, we use the definition of $\Delta T_{wm} = \Delta T_F \, \Delta\theta_w$, where ΔT_F is a decreasing function of Pe, and $\Delta\theta_w$ is an increasing function of Pe. Therefore, the numerical calculations show that the decrease of ΔT_F is a dominant factor if q is also decreased, which is in concordance with the relationship (6). Figure 5, shows similar behavior for the corresponding nondimensional average temperature differences, $(\bar{T}_w - T_\infty)/(qHL/\lambda)$, plotted as a function of the Peclet number Pe. Also, the comparison between numerical and analytical results shows a good agreement.

Finally, the above results for the cooling mechanism of the heated strip are completed with the estimation of the heat transfer. As an illustration, in Fig. 6 we show the reduced local Nusselt number $Nu_r = Nu_\chi/[(H/L)^2(1/6Pe)]^{-1/3}$ as a function of the coordinate χ only for the limiting cases of $\alpha = 0$ and $\alpha \gg 1$ (here, we define the local Nusselt number in terms of nondimensional variables, as $Nu_\chi = [(H/L)^2(1/6Pe)]^{-1/3}(\chi^{2/3}/\theta_w).(\partial\theta/\partial\zeta)|_{\zeta=0}$). In the first case, with $\alpha = 0$, we obtain $Nu_r = [B(1/3, 2/3)/3]\chi^{2/3} = 1.2092\chi^{2/3}$, whereas with $\alpha \gg 1$, $Nu_r = \chi^{2/3}$. We have selected extreme values in α to obtain the differences between the two solutions. In particular, this comparison can be illustrated very well by defining the reduced average Nusselt number as

$$Nu = \int_0^1 Nu_r \, \mathrm{d}\chi$$

For each case, we obtain that $Nu(\alpha = 0) = 0.7255$ and $Nu(\alpha \gg 1) = 0.6$, yielding a difference percentage of 17.33%. Therefore, for the limit of $\alpha/\varepsilon^2 \gg 1$, the cooling mechanism shows a clear dependence on assumed values of α.

VI. Conclusions

In the present work, we have carried out an analytical and numerical analysis to study the mechanism of cooling of an electronic chip embedded in a rectangular channel. Inside this channel is circulating a laminar flow, which permits to transfer heat from the heating chip to the fluid. Because the substrate was considered perfectly adiabatic, the heat transfer rate only occurs between the strip and the cooling flow. Here, this cooling mechanism is modulated by the introduction of the nondimensional parameter α, which reflects the conjugate character of the problem. Basically, in terms of this nondimensional parameter α, the thermal performance and characteristics of this typical device are well defined.

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